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A Novel Generalized Divergence Measure of Fuzzy Sets

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ABSTRACT

Aim: The aim of the paper is to introduce a generalized measure of discrimination in fuzzy environment.

Methodology: To achieve the goal of this paper a parametric generalization of Hellinger's fuzzy divergence measure is studied along with the proof of its validity.

Results: A particular case and some important properties are discussed in detail of the proposed generalized Hellinger's fuzzy divergence measure.

Conclusion: Generalized Hellinger's fuzzy divergence measure is valid measure of fuzzy divergence.

Key Words: Fuzzy sets, Fuzzy entropy, Fuzzy divergence measure, Generalized Fuzzy Divergence Measures

INTRODUCTION

Shannon [1] was first to use the word "entropy" to measure an uncertain degree of the randomness in a probability distribution. The theory of fuzzy sets and fuzzy logic developed by Zadeh [2] has been used to form ambiguity, lack of information and uncertainty inherent in the human decision making process. It has achieved a great success in various areas such as multi-criteria decision making (MCDM), logical programming, pattern recognition, medical diagnosis and so on. Zadeh [3] introduced fuzzy entropy as an significant concept for measuring fuzzy information.

Thereafter, a number of researchers made study on the fuzzy theory and find their applications in different areas. For example, some new information and divergence measures and their applications in different areas have been proposed in literature [4-22].

Although many information measures between fuzzy sets has been emerging in the last decades. Still, there is a need to define quantitative information measures for imprecision, discrimination, distance, etc. over fuzzy sets with their practical applications. In literature the Hellinger's measure of discrimination was firstly introduced by Hellinger [23]. Here we propose a new parametric generalized Hellinger's divergence measure in fuzzy environment which provides a flexible approach to further leverage of choice to the user. It may be observed that the potential, strength and efficiency of this new generalized Hellinger's fuzzy divergence measure exist in its properties.

Methodology section we recall and discuss some well-known concepts and the notions related to fuzzy set theory. In results section we introduce a generalized Hellinger's fuzzy divergence measure with the proof of validity. Some interesting properties of the proposed measure between different fuzzy sets are studied drawn in discussion section. Final section concludes the paper.

METHODOLOGY

We now review a number of well-known concepts and definitions related with the theory of fuzzy sets. The uncertainty and vagueness in the environment can be easily handled by fuzzy sets.

Definition 1. *Fuzzy Set(FS)* [2]: A fuzzy set (FS) X on a universe of discourse $U = (u_1, u_2, \ldots, u_n)$ having the membership function $\mu_X : U \to [0,1]$ as follows:

$$X = \left\{ \left\langle a, \mu_X(a) \right\rangle / a \in U \right\}$$

The membership value $\mu_X(a)$ describes the degree of the belongingness of $a \in U$ in *X*. When $\mu_X(a)$ is valued in {0, 1}, it is the characteristic function of a crisp (i.e., non-fuzzy) set.

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The fuzzy divergence measure can be defined as the difference between two fuzzy sets.

Bhandari and Pal [24] established the fuzzy divergence measure analogous to Kullback and Leibler [25] divergence measure, as

$$I(A:B)=\sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}+(1-\mu_A(x_i))\log\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right] \qquad (1)$$

with the conditions:

$I(A:B)=0$ *if* $A=B$

$I(A:B)=0$ *if* $A=B$

$I(A:B)$ is a convex function of $\mu_A(x_i)$.

RESULTS

We now propose a generalized measure of Hellinger's divergence between two fuzzy sets A and B defined in a universe of discourse $U=\{a_1,a_2,...,a_n\}$ having membership values $\mu_X(a_i),\mu_Y(a_i)\in(0,1)$ corresponding to Taneja [26] generalized Hellinger's discrimination measure given by

$$h_t(X,Y)=\sum_{i=1}^{n}\left[\frac{\left(\sqrt{\mu_X(a_i)}-\sqrt{\mu_Y(a_i)}\right)^{2(t+1)}}{\left(\sqrt{\mu_X(a_i)\mu_Y(a_i)}\right)^t}+\frac{\left(\sqrt{1-\mu_X(a_i)}-\sqrt{1-\mu_Y(a_i)}\right)^{2(t+1)}}{\left(\sqrt{(1-\mu_X(a_i))(1-\mu_Y(a_i))}\right)^t}\right], t\in N \qquad (2)$$

Theorem 1 $h_t(X,Y)$ is the valid divergence measure of fuzzy sets.

Proof: It is clear from (2) that

(i) $h_t(X,Y)\geq 0$

(ii) $h_t(X,Y)=0$ if $\mu_X=\mu_Y$

(iii) We now check the convexity of $h_t(X,Y)$.

$$\frac{\partial h_t(X,Y)}{\partial\mu_X(a_i)}$$

$$=\frac{(2t+2)\left(\sqrt{\mu_X(a_i)}-\sqrt{\mu_Y(a_i)}\right)^{(2t+1)}}{(\mu_X(a_i)\mu_Y(a_i))^{t/2}}\frac{1}{2\sqrt{\mu_X(a_i)}}+\left(-\frac{t}{2}\right)\frac{\left(\sqrt{\mu_X(a_i)}-\sqrt{\mu_Y(a_i)}\right)^{(2t+2)}}{(\mu_X(a_i)\mu_Y(a_i))^{-\frac{t}{2}-1}}\mu_Y(a_i)$$

$$+2(t+1)\frac{\left(\sqrt{1-\mu_X(a_i)}-\sqrt{1-\mu_Y(a_i)}\right)^{(2t+1)}}{\left(\sqrt{(1-\mu_X(a_i))(1-\mu_Y(a_i))}\right)^t}\frac{1}{2\sqrt{1-\mu_X(a_i)}}(-1)-\frac{t}{2}\frac{\left(\sqrt{1-\mu_X(a_i)}-\sqrt{1-\mu_Y(a_i)}\right)^{(2t+2)}}{((1-\mu_X(a_i))(1-\mu_Y(a_i)))^{\frac{t}{2}+1}}[-(1-\mu_Y(a_i))]$$

$$=(t+1)\left[\frac{\left(\sqrt{\mu_X(a_i)}-\sqrt{\mu_Y(a_i)}\right)^{(2t+1)}}{\sqrt{\mu_X(a_i)}\left(\sqrt{\mu_X(a_i)\mu_Y(a_i)}\right)^t}-\frac{\left(\sqrt{1-\mu_X(a_i)}-\sqrt{1-\mu_Y(a_i)}\right)^{(2t+1)}}{\sqrt{1-\mu_X(a_i)}\left(\sqrt{(1-\mu_X(a_i))(1-\mu_Y(a_i))}\right)^t}\right]$$

$$-\frac{t}{2}\left[\frac{\left(\sqrt{\mu_X(a_i)}-\sqrt{\mu_Y(a_i)}\right)^{(2t+2)}}{\mu_X(a_i)\left(\sqrt{\mu_X(a_i)\mu_Y(a_i)}\right)^t}-\frac{\left(\sqrt{1-\mu_X(a_i)}-\sqrt{1-\mu_Y(a_i)}\right)^{(2t+2)}}{(1-\mu_X(a_i))\left(\sqrt{(1-\mu_X(a_i))(1-\mu_Y(a_i))}\right)^t}\right]$$

$$\frac{\partial^2 h_t(X,Y)}{\partial\mu_X^2(a_i)}=\frac{(t+1)(2t+1)}{2}\left[\frac{\left(\sqrt{\mu_X(a_i)}-\sqrt{\mu_Y(a_i)}\right)^{2t}}{\mu_X(a_i)\left(\sqrt{\mu_X(a_i)\mu_Y(a_i)}\right)^t}+\frac{\left(\sqrt{1-\mu_X(a_i)}-\sqrt{1-\mu_Y(a_i)}\right)^{2t}}{(1-\mu_X(a_i))\left(\sqrt{(1-\mu_X(a_i))(1-\mu_Y(a_i))}\right)^t}\right]$$

$$-\frac{(t+1)(2t+1)}{2}\left[\frac{\left(\sqrt{\mu_X(a_i)}-\sqrt{\mu_Y(a_i)}\right)^{2t+1}}{\mu_X^{3/2}(a_i)\left(\sqrt{\mu_X(a_i)\mu_Y(a_i)}\right)^t}+\frac{\left(\sqrt{1-\mu_X(a_i)}-\sqrt{1-\mu_Y(a_i)}\right)^{2t+1}}{(1-\mu_X(a_i))^{3/2}\left(\sqrt{(1-\mu_X(a_i))(1-\mu_Y(a_i))}\right)^t}\right]$$

$$+\frac{t(t+2)}{4}\left[\frac{\left(\sqrt{\mu_X(a_i)}-\sqrt{\mu_Y(a_i)}\right)^{2t+2}}{\mu_X^2(a_i)\left(\sqrt{\mu_X(a_i)\mu_Y(a_i)}\right)^t}+\frac{\left(\sqrt{1-\mu_X(a_i)}-\sqrt{1-\mu_Y(a_i)}\right)^{2t+2}}{(1-\mu_X(a_i))^2\left(\sqrt{(1-\mu_X(a_i))(1-\mu_Y(a_i))}\right)^t}\right]>0,$$

$t\in N$.

Similarly, $t\in N$, $t\in N$.

Thus $h_t(X,Y)$ is a convex function of fuzzy sets X and Y and hence in view of the definition of fuzzy divergence measure of Bhandari and Pal [2] provided in Section 2, $h_t(X,Y)$ is a valid measure of fuzzy divergence.

Particular Case: For $2h(X,Y)$, $2h(X,Y)$ reduces to $2h(X,Y)$ where $h(X,Y)$ is fuzzy Hellinger's divergence measure.

DISCUSSION

We now provide some more properties of the proposed generalized fuzzy divergence measure (2) in the following theorems. While proving these theorems we consider the distribution of U in to two parts U_1 and U_2 as

$$U_1=\{a/a\in U,\mu_X(a_i)\geq\mu_Y(a_i)\} \qquad (3)$$

and $$h_t(X\cup Y,X)+h_t(X\cap Y,X)=h_t(X,Y) \qquad (4)$$

Theorem 2

(a) $h_t(X\cup Y,X)+h_t(X\cap Y,X)=h_t(X,Y)$

(b) $h_t(\overline{X\cup Y},\overline{X\cap Y})=h_t(\overline{X}\cap\overline{Y},\overline{X}\cup\overline{Y})=h_t(X,Y)$

(c) $h_t(X,\overline{X})=h_t(\overline{X},X)$.

Theorem 3

(a) $h_t(X\cup Y,Z)+h_t(X\cap Y,Z)=h_t(X,Z)+h_t(Y,Z)$.

(b) $h_t(X,X\cap Y)=h_t(Y,X\cup Y)$.

(c) $h_t(X,X\cap Y)=h_t(Y,X\cup Y)$.

Theorem 4

(a) $h_t(X\cup Y,X\cap Y)=h_t(X,Y)$.

(b) $h_t(X,\overline{Y})=h_t(\overline{X},Y)$.

(c) $h_t(X,\overline{Y})=h_t(\overline{X},Y)$.

(d) $h_t(X,Y)+h_t(\overline{X},Y)=h_t(\overline{X},\overline{Y})+h_t(X,\overline{Y})$.

Proof: Proof of above theorems follows from (3) and (4).

CONCLUSION

In the present paper we have obtained generalized information measure of discrimination in fuzzy setting. For this we have proposed and validated the generalized Hellinger's measure of fuzzy divergence. A particular case and a number of the interesting efficient properties of this generalized divergence measure are proven. Finally, we observe that the presence of the parameters in the proposed measure provides a greater flexibility in applications.

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